271. Proposed by E. B. ESCOTT, Ann Arbor, Mich.

In the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2\frac{d^5y}{dx^5}-5\left(\frac{d^2y}{dx^2}\right)\frac{d^3y}{dx^3}\cdot\frac{d^4y}{dx^4}+\tfrac{40}{9}\left(\frac{d^3y}{dx^3}\right)^3=0,$$

show that there is an integrating factor of the form $\left(\frac{d^2y}{dx^2}\right)^n$, and integrate the equation.

Solution by the PROPOSER, and LEVI S. SHIVELY, Mt. Morris College, Mt. Morris, Ill.

When the equation is multiplied by $(d^2y/dx^2)^n$, if it is an exact equation, its first integral is

$$\left(\frac{d^2y}{dx^2}\right)^{n+2}\frac{d^4y}{dx^4}+\frac{40}{9(n+1)}\left(\frac{d^2y}{dx^2}\right)^{n+1}\left(\frac{d^3y}{dx^3}\right)^2=c_1 \ldots (2).$$

Differentiating this equation and comparing with original equation, we find that $n=-\frac{11}{3}$ and $-\frac{13}{3}$.

In (2), put $n==\frac{11}{3}$. The equation is exact and its integral is

$$\left(\frac{d^2y}{dx^2}\right)^{-\frac{5}{3}}\frac{d^3y}{dx^3}=c_1x+c_2.$$

Integrating again,

$$-\tfrac{3}{2}\left(\frac{d^2y}{dx^2}\right)^{-\frac{2}{3}}=\frac{c_1}{2}x^2+c_2x+c_3.$$

Solving for d^2y/dx^2, and changing the constants,

$$\frac{d^2y}{dx^2}=\frac{1}{(c_1x^2+2c_2x+c_3)^{\frac{3}{2}}}.$$

Integrating twice, we have

$$y=\frac{1}{c_1c_3-c_2^2}\sqrt{(c_1x^2+2c_2x+c_3)}+c_4x+c_5.$$

Also solved by G. B. M. Zerr, and V. M. Spunar.

272. Proposed by CLARENCE OHLENDORF, Chicago, Ill.

Find $\int\log_e\tan^{-1}xdx$.

Solution by J. SCHEFFER, A. M., Hagerstown, Md., and C. N. SCHMALL, New York City.

Putting $x=\tan y$, we have

$$\int\log\tan^{-1}xdx=\int\log y\frac{dy}{\cos^2y}=\int\log y\sec^2 ydy=\log y-\tan y-\int\frac{\tan ydy}{y}.$$

Substituting for tan$y$ its series, and integrating, we get

$$\int \frac{\tan y dy}{y} = \frac{2^2(2^2-1)}{2!} B_1 y + \frac{2^4(2^4-1)}{4!} B_3 \frac{y^3}{3} + \frac{2^6(2^6-1)}{6!} B_5 \frac{y^5}{5},$$

where B_1, B_3, B_5, ... are Bernoulli's numbers.

Therefore the given integral is

$$x \log \tan^{-1}x - \left[\frac{2^2(2^2-1)}{2!} B_1 \tan^{-1}x + \frac{2^4(2^4-1)}{4!} B_3 \frac{(\tan^{-1}x)^3}{3} + \ldots\right].$$

Also solved by G. B. M. Zerr, V. M. Spunar, and J. E. Sanders.

273. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

On one side of a circular pond a feet in radius is a duck. On the diametrically opposite side of the pond is a dog. Both begin to swim at the same time, the duck swimming around the circumference of the pond at the rate of m feet a minute, the dog swimming directly towards the duck at the rate of n feet per minute. How far will the dog swim in overtaking the duck?

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

If the line joining the positions of the dog and the duck is tangent to the dog's path, the following is the solution.

Let A, B be the starting points of dog and duck; O, the center of the pond; P, R, corresponding positions of the dog and duck. Then the tangent t, from P must pass through R. Let the duck swim to S and the dog to Q. Draw OF, PH perpendicular to AB, PI, QD perpendicular to OF, QC perpendicular to PI, and RE perpendicular to QS, produced.

Let $AB=2a$, $AP=\sigma$, $\angle RAB=\phi$, $OD=x$, $n/m=b$. Then $PR=d\sigma$, $\angle RAS=d\phi$, $DI=QC=dx$.

Now $\sigma=2ab\phi$; $\therefore d\sigma/2ad\phi=b$.

In the limit the triangles PQC and RES are similar.

$\therefore d\sigma : dx=2ad\phi : ES$. $\therefore ES=(2ad\phi/d\sigma)dx=dx/b$.

The tangent t has negative increments at both ends (PQ and ES).

$\therefore dt=-d\sigma-dx/b$, or $t=C-\sigma-x/b$. When $t=2a$, $\sigma=0$, $x=0$.

$\therefore C=2a$. $\therefore t=2a-2ab\phi-x/b$. When $t=0$, $x=2a\sin\phi\cos\phi$.

$\therefore b=b^2\phi+\sin\phi\cos\phi=(b^2+1)\phi-\frac{2}{3}\phi^3+\frac{2}{15}\phi^5-\frac{4}{315}\phi^7+\ldots$

$$Q=\frac{b}{b^2+1}+\frac{2b^4}{3(b^2+1)^4}+\frac{2b^5(9-b^2)}{15(b^2+1)^7}+\frac{4(225+b^4-54b^2)b^7}{315(b^2+1)^{10}}+\ldots$$

by reversion of series.